Chapter 13

Adaptive Beamforming

We have already considered deterministic beamformers for such applications as pencil beam arrays and arrays with controlled sidelobes. Beamformers can also be developed for other design goals, including interference or jammer suppression, target location, imaging, and shaped patterns for multiuser communications. For these applications, the ideal beamformer is adaptive, so that the radiation pattern can be adjusted to suppress noise and interference and maximize sensitivity to the desired signal. In principle, the beamformer could react to changes in the signal and noise voltages at the array output, but the variations in these voltages may be too rapid to be tracked by the signal processing hardware. A better solution is to compute array output voltage correlation matrices, in order to characterize the longer term stochastic properties of the propagation environment.

These considerations motivate the study of statistically optimal and adaptive beamforming algorithms that seek to optimize some measure of system performance such as average SNR using estimates of the statistics of the environment. We will analyze adaptive beamformers from a receive point of view, although many of the concepts to be developed can also be applied to a transmitter given feedback from a receiver or other sensor that measures properties of the propagation environment.

13.1 Signal Steering Vectors

We will first review the model for array output voltages developed in Section 8.1. The open circuit voltages $v_{\mathrm{oc},n}$ at the terminals of a receiving antenna can be found from (8.9). These voltages as a vector $\mathbf{v}_{\mathrm{oc}}$ are transformed through a system transform matrix $\mathbf{Q}$ as defined in (8.4) to a vector of voltages $\mathbf{v}$ at the outputs of the signal chains connected to each array element. The contribution to the array output voltages due to the signal of interest is

$$\mathbf{v}_{\mathrm{sig}} = \mathbf{Q}\mathbf{v}_{\mathrm{oc,sig}} \tag{13.1}$$

In terms of the embedded element radiation patterns,

$$\mathbf{v}_{\mathrm{sig}} = c_1 E^{\mathrm{inc}} \mathbf{Q}\mathbf{E}_p \tag{13.2}$$

where the constant c_1 is defined in (8.11), $\mathbf{E}_p$ is a vector of inner products of the incident field polarization with the array embedded element patterns defined in (7.18), and E^{inc} is the amplitude of the incident field representing the signal of interest at the origin of the coordinate system in which the receiving array embedded element radiation patterns are defined. In terms of these quantities, the signal of interest portion of the beamformer output voltage can be expressed as

$$v_{\mathrm{out}} = \mathbf{w}^H \mathbf{v}_{\mathrm{sig}} \tag{13.3}$$

where $\mathbf{w}$ is a beamformer weight vector.

When working with adaptive arrays, it is convenient to define the steering vector

$$\mathbf{d} = \sqrt{N}\|\mathbf{v}_{\mathrm{sig}}\|^{-1}\mathbf{v}_{\mathrm{sig}} \tag{13.4}$$

This vector is proportional to $\mathbf{v}_{\mathrm{sig}}$, but is scaled so that it is independent of the incident plane wave intensity. By convention, the normalization of the steering vector is such that

$$\|\mathbf{d}\|^2 = \mathbf{d}^H\mathbf{d} = N \tag{13.5}$$

where N is the number of array elements.

If the signal amplitude is a function of time, then the array output voltage vector can be written using the steering vector as

$$\mathbf{v}_{\mathrm{sig}}s(t) = s(t)\mathbf{d} \tag{13.6}$$

where $s(t)$ is a phasor or complex baseband representation of the signal waveform. The array output voltage correlation matrix can be written in the form

$$\mathbf{R}_{\mathrm{sig}} = \sigma_{\mathrm{s}}^2\mathbf{d}\mathbf{d}^H \tag{13.7}$$

where $\sigma_{\mathrm{s}}^2 = \mathrm{E}[|s(t)|^2]$. This is an alternate form of the rank one signal correlation matrix given in (8.24). Since the correlation matrix $\mathbf{R}_{\mathrm{sig}}$ has only one nonzero eigenvalue equal to $\sigma_{\mathrm{s}}^2 N$, and the matrix trace is equal to the sum of the eigenvalues, we have $\mathrm{tr}\,\mathbf{R}_{\mathrm{sig}} = \sigma_{\mathrm{s}}^2 N$. By analogy with (12.17) for a transmitting array, it can be seen that σ_{s}^2 is proportional to the average available signal power at the receiver outputs ports.

The steering vector $\mathbf{d}$ takes on a particularly simple form if mutual coupling between array elements is neglected. If we assume that the array elements are identical and neglect array edge effects, the open circuit loaded element patterns become

$$\overline{E}_n(\overline{r}) = e^{j\overline{k}\cdot\overline{r}_n}\overline{E} \tag{13.8}$$

where $\overline{E}$ is the field radiated by an element located at the origin, $\overline{r}_n$ is the location of the nth array element, and $\overline{k}$ is a wavevector pointing in the direction of the source. The wavevector can be expressed as

$$\overline{k} = k\hat{k} \tag{13.9}$$

in terms of the unit vector $\hat{k}$, which is in the direction of arrival (DOA) of the incident plane wave. Neglecting mutual coupling for identical elements with identical loads also means that the receiver output voltages are proportional to the element open circuit voltages, since $\mathbf{Q}$ becomes a scaled identity matrix. Under these assumptions, the beamformer output is

$$v_{\mathrm{out}} = \underbrace{\frac{4\pi jre^{-jkr}}{\omega\mu}E^{\mathrm{inc}}\hat{p}\cdot\overline{E}(\overline{r})}_{c_3}\sum_n w_n^* e^{jk\hat{k}\cdot\overline{r}_n} \tag{13.10}$$

From this expression, it can be seen that the steering vector has components

$$d_n = e^{jk\hat{k}\cdot\overline{r}_n} \tag{13.11}$$

The beam output voltage is

$$v_{\mathrm{out}} = c_3\mathbf{w}^H\mathbf{d} \tag{13.12}$$

Since $k = \omega/c$, we can interpret the exponent $k\hat{k}\cdot\overline{r}_n = \omega\hat{k}\cdot\overline{r}_n/c = \omega\tau_n$ in terms of the time delay $\tau_n = \hat{k}\cdot\overline{r}_n/c$ of the signal at the nth array element relative to the signal at the origin. This allows us to view the beamformer as a discrete filter with taps w_n^* at the delays τ_n.

13.2 Multiple Sidelobe Canceler (MSC)

The multiple sidelobe canceling architecture was one of the first adaptive beamforming methods developed for array antennas. The goal is to reject an interfering signal while receiving a signal of interest. For this system we have a primary channel and N auxiliary channels, where the primary channel is typically a single antenna and the auxiliary channels are an array. The auxiliary signals are combined with a beamformer and then subtracted from the primary signal.

If we denote the output of the primary channel as x_p, and the auxiliary array outputs as $\mathbf{x}$, then the overall system output signal is

$$x_{\text{out}} = x_p - \mathbf{w}^H \mathbf{x} \tag{13.13}$$

The goal is to design the beamformer weights $\mathbf{w}$ to reject the undesired interferer. In the absence of the desired signal, we want $x_{\text{out}} = 0$. If we measure the outputs in the absence of the desired signal, then we can design the beamformer weights according to

$$\mathbf{w} = \underset{\mathbf{w}}{\operatorname{argmin}}\, \mathrm{E}[|x_p - \mathbf{w}^H \mathbf{x}|^2] \tag{13.14}$$

Expanding the expectation leads to the condition

$$\mathrm{E}[\mathbf{x}\mathbf{x}^H]\mathbf{w} = \mathrm{E}[\mathbf{x}x_p^*] \tag{13.15}$$

or

$$\mathbf{w} = \mathbf{R}_{\mathbf{xx}}^{-1}\mathbf{R}_{\mathbf{x}x_p} \tag{13.16}$$

where $\mathbf{R}_{\mathbf{x}x_p}$ is a column vector. This beamformer places nulls of the overall antenna pattern on the interfering signal.

The difficulty with MSC is that the desired signal must be absent from the auxiliary outputs when computing the weights or small in amplitude relative to noise and interference, which means that MSC is effective for very weak desired signals. Another limitation is that MSC does not steer the main beam towards the desired signal.

13.3 Minimum Mean Squared Error (MMSE)

The minimum mean squared error (MMSE) beamformer leads to a set of array beamformer weights that minimizes the difference between the correlation statistics across the array for a desired received signal and the array output. MMSE is based on the concept from signals and systems analysis that the best approximation to a desired signal is obtained when the error is orthogonal to the signal, which is known as the orthogonality principle.

If a plane wave carrying a desired signal $s(t)$ arrives at an array along with other waves carrying noise and interference, the MMSE beamformer is defined by the minimization problem

$$\mathbf{w} = \underset{\mathbf{w}}{\operatorname{argmin}}\; \mathrm{E}[|s - \mathbf{w}^H \mathbf{x}|^2] \tag{13.17}$$

where $\mathbf{x}$ is a vector of array output voltages. The quantity inside the square brackets is the error signal, or the difference between the desired signal and the beamformer output.

Using the orthogonality principle, it can be shown that with the minimizing beamformer weight vector, the error signal is statistically orthogonal to the received signal with optimal beamformer weights. This leads to the condition

$$\begin{aligned} 0 &= \mathrm{E}[\mathbf{x}^H(s - \mathbf{w}^H\mathbf{x})] \\ &= \mathrm{E}[\mathbf{x}^H s - \mathbf{x}^H\mathbf{w}^H\mathbf{x}] \\ &= \mathrm{E}[\mathbf{x}s^* - \mathbf{x}\mathbf{x}^H\mathbf{w}] \end{aligned}$$

Because expectation is linear, the expectations of the two terms must be equal, and we have

$$\mathrm{E}[\mathbf{x}s^*] = \mathrm{E}[\mathbf{x}\mathbf{x}^H]\mathbf{w} \tag{13.18}$$

In terms of correlation matrices, this leads to a linear system that can be solved for the beamformer weight vector,

$$\mathbf{R}_{\mathbf{x}s} = \mathbf{R}_{\mathbf{xx}}\mathbf{w} \tag{13.19}$$

The solution is

$$\mathbf{w} = \mathbf{R}_{\mathbf{xx}}^{-1}\mathbf{R}_{\mathbf{x}s} \tag{13.20}$$

The MMSE beamformer weight vector is therefore obtained from the array output correlation matrix and the cross-correlation of the signal of interest and the array outputs.

This beamformer has a nice statistical optimality property, in that it minimizes error at the beamformer output, but calculating $\mathbf{w}$ requires that we know the array output covariance and the cross-covariance of the desired signal and the array outputs. The covariances can be estimated using a model or computed using signal processing on the array outputs and desired signal. If the SNR is low and the noise at the array outputs is IID, then $\mathbf{R}_{\mathbf{xx}}$ is approximately a scaled identity, and can be ignored in (13.20). Assuming that the signal of interest and the noise are uncorrelated, the column vector $\mathbf{R}_{\mathbf{x}s}$ is proportional to the signal steering vector (13.4), and the MMSE beamformer reduces to the conjugate field match beamformer (7.39).

13.4 Maximum SNR Beamformer

We have already covered many aspects of SNR at the output of a beamforming array. The last remaining topic is to find the beamformer weights that maximize SNR, or the max-SNR beamformer. We will find that the although the max-SNR beamformer is defined in an entirely different framework, it is closely related to the maximum directivity beamformer (7.38).

The SNR at the output of a beamformer is

$$\mathrm{SNR} = \frac{\mathbf{w}^H\mathbf{R}_\mathrm{s}\mathbf{w}}{\mathbf{w}^H\mathbf{R}_\mathrm{n}\mathbf{w}} \tag{13.21}$$

where $\mathbf{R}_\mathrm{s}$ and $\mathbf{R}_\mathrm{n}$ are the signal and noise covariance matrices, respectively. The max-SNR beamformer is defined by

$$\mathbf{w} = \underset{\mathbf{w}}{\mathrm{argmax}}\ \frac{\mathbf{w}^H\mathbf{R}_\mathrm{s}\mathbf{w}}{\mathbf{w}^H\mathbf{R}_\mathrm{n}\mathbf{w}} \tag{13.22}$$

We have already shown in Section 7.6.2 that maximizing a ratio of quadratic forms leads to the generalized eigenvalue problem

$$\mathbf{R}_\mathrm{s}\mathbf{w} = \lambda_\mathrm{max}\mathbf{R}_\mathrm{n}\mathbf{w} \tag{13.23}$$

where λ_max is the largest generalized eigenvalue.

If $\mathbf{R}_\mathrm{s}$ is a rank one matrix of the form $\sigma_\mathrm{s}^2\mathbf{d}\mathbf{d}^H$, corresponding to a single point source, then

$$\mathbf{w} = \mathbf{R}_\mathrm{n}^{-1}\mathbf{d} \tag{13.24}$$

The attained value of the SNR is

$$\begin{aligned}\mathrm{SNR}_\mathrm{max} &= \frac{(\mathbf{R}_\mathrm{n}^{-1}\mathbf{d})^H\sigma_\mathrm{s}^2\mathbf{d}\mathbf{d}^H\mathbf{R}_\mathrm{n}^{-1}\mathbf{d}}{(\mathbf{R}_\mathrm{n}^{-1}\mathbf{d})^H\mathbf{R}_\mathrm{n}\mathbf{R}_\mathrm{n}^{-1}\mathbf{d}}\\ &= \sigma_\mathrm{s}^2\mathbf{d}^H\mathbf{R}_\mathrm{n}^{-1}\mathbf{d}\end{aligned} \tag{13.25}$$

This expression is essentially the ratio of the signal power σ_s^2 to the noise power received by the beamformer. The noise correlation matrix is inverted, so that larger noise power corresponds to a smaller value for the elements of $\mathbf{R}_n^{-1}$, which makes it clear that (13.25) decreases as the noise level becomes stronger.

If the noise at the array outputs consists only of spatially isotropic thermal noise, then according to (8.30), $\mathbf{R}_n \sim \mathbf{A}$, where $\mathbf{A}$ is the array embedded element pattern overlap matrix. In this case, the max-SNR beamformer is equivalent to the maximum directivity beamformer (7.38). Physically, this can be understood by observing that in an isotropic noise environment, the equivalent temperature of the external noise is constant and is independent of the beamformer weights. The only degree of freedom that can be exploited to increase SNR is to receive as much signal as possible, which is precisely what the maximum directivity beamformer does.

For more complex noise models, in order to apply the max-SNR beamformer, the signal steering vector and the noise correlation matrix must be measured. This process is sometimes referred to as array calibration. If the phased array is a feed on an astronomical dish antenna, for example, the noise correlation matrix can be measured by steering the dish so that the main beam is pointed to an area of sky with no strong stars or other radio sources. The signal steering vector can be measured by pointing the dish to a bright calibrator source such as an intense radio galaxy. Multiple beams can be formed to produce a multipixel image by steering the dish so that the calibrator source is in various locations relative to the boresight direction of the dish antenna. The max-SNR weights then provide a set of beamformer coefficients that can be used to form a high sensitivity beam to observe and create images of astronomical sources of interest.

13.5 Linearly Constrained Minimum Variance Beamformer (LCMV)

The MMSE and max-SNR beaformers result from unconstrained optimization problems. In some cases, we wish to maximize the received signal subject to some additional constraint, such as a given level of response to the signal of interest, a controlled beamshape, a prescribed null on an interfering source, or another type of pattern design goal.

The basic linearly constrained minimum variance beamformer (LCMV) includes a constraint to ensure that the desired signal is received at a specified complex voltage level. Subject to this constraint, we minimize the total variance of the beamformer output, which means that we minimize noise power received from other directions. The LCMV beamformer is defined by

$$\mathbf{w} = \operatorname{argmin} \mathbf{w}^H \mathbf{R}_{\mathbf{xx}} \mathbf{w}, \quad \text{subject to } \mathbf{w}^H \mathbf{d} = g \tag{13.26}$$

where $\mathbf{d}$ is the signal steering vector. Using the method of Lagrange multipliers, the solution

$$\mathbf{w} = \frac{g^*}{\mathbf{d}^H \mathbf{R}_{\mathbf{xx}} \mathbf{d}} \mathbf{R}_{\mathbf{xx}}^{-1} \mathbf{d} \tag{13.27}$$

can be obtained. This is also known as Capon's beamformer.

In the case of exactly known signal and noise correlation matrices and a rank one signal of interest, it can be shown that (13.27) gives the same beamformer weight vector as the max-SNR beamformer. By extending this derivation to a vector of constraints, LCMV can be used to place a null on a fixed interferer ($g = 0$), create multiple main beams to receive multiple desired signals of interest, or obtain a desired beam shape. If LCMV is extended to multiple constraints, there is always a tradeoff between SNR and other design goals. Since the max-SNR beamformer achieves the best possible SNR, any other nontrivially different set of beamformer weights realizes a lower SNR, but may be better than the max-SNR beamformer in other respects.

13.6 Subspace Projection

If the design goal is to place nulls on one or more intefering signals, the method of subspace projection can be used. If the steering vector associated with an interfering signal is $\mathbf{d}_i$, then we can form the projection operator

$$\mathbf{P} = \mathbf{I} - \tfrac{1}{N}\mathbf{d}_i\mathbf{d}_i^H \tag{13.28}$$

where the scale factor normalizes the vectors in the rank one term to unit length. A beamformer weight vector from another algorithm such as max-SNR can be transformed into a new beamformer according to

$$\mathbf{w}_{\mathrm{SP}} = \mathbf{P}\mathbf{w} \tag{13.29}$$

It is easy to see that if this beamformer weight vector is applied to a rank one signal response correlation matrix due to a signal arriving with steering vector $\mathbf{d}_i$, the response of the beamformer is zero. It is also easy to extend this method to the case of multiple interferers.

Since the subspace projection method modifies the original beamformer weight vector, the SNR achieved is in general modified. If the base beamformer is max-SNR, then the SNR is reduced. If the interferer is included in the max-SNR beamformer, then the max-SNR beamformer already maximizes the ratio of signal to interference and noise. The motivation for using the SP method to further reduce the interferer is that in some cases, the temporal properties of the interferer makes it more harmful to the signal of interest detection process than thermal noise. In the exact case, SP reduces the interferer to zero, but if there is error in the steering vector estimation, the pattern null is not identically zero at the interferer arrival angle. Error is caused by correlation matrix estimation error, interferer motion, and other effects. To a degree, these sources of error can be compensated for by using more sophisticated array signal processing [10].

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